Nonintegrable Dynamics and the Breakdown of Stationary Action

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Abstract

As long-cherished postulate of theoretical physics, the *principle of stationary action* (SAP) defines the basis of classical mechanics and field theory. We argue here that SAP is overturned in dynamic regimes where sensitivity to initial conditions cannot be ignored. Attempts are made to bridge the gap between Hamiltonian chaos and effective field theory.

Key words: stationary action, decoherence, Hamiltonian chaos, KAM theorem, Arnold diffusion, action quantization, effective field theory.

1. Introduction

As it is known, integrable systems form the backbone of classical and quantum field theory. A Hamiltonian (conservative) system with N degrees of freedom is integrable if it has N independent commuting constants of motion. An important attribute of this class of systems is that all interactions can be eliminated by appropriate canonical transformations. Integrability implies the existence of periodic or quasi-periodic tori in phase-space, a property that can be extended to dissipative systems [1-3, 12].

Nature shows, however, that most interacting Hamiltonian systems are *nonintegrable* and their long-term evolution chaotic. The primary mechanism explaining the onset of Hamiltonian chaos is the Kolmogorov-Arnold-Moser (KAM) theorem, which is the perturbation theory of quasi-periodic tori applied to nearly-integrable Hamiltonian systems.

In the context of this work, we take nonintegrability to arise either from *sensitivity to initial conditions* or *undamped perturbations* outside equilibrium. While sensitivity to initial conditions describes transition to chaos via positive Lyapunov exponents, undamped perturbations generate chaos via the progressive collapse of quasi-periodic tori, fragmentation of phase-space and the emergence of fractal spacetime [4-5].

As conjectured in several publications, the mechanism of *decoherence* - the loss of phase information and the entropy surge in open systems – comes into play beyond the Standard Model scale and favors the transition from quantum to classical behavior. A reasonable expectation is that deep Terascale physics falls outside thermodynamic equilibrium and, in doing so, it replicates the attributes of Hamiltonian chaos [6-9].

The object of this work is to argue that sensitivity to initial conditions is bound to overturn SAP and, on account of decoherence, to bridge the gap between Hamiltonian chaos and the foundations of effective field theory.

The paper is formatted in the following way: next section contains a brief introduction to SAP in classical field theory, with emphasis on electrodynamics and General Relativity. The breakdown of SAP due to sensitive dependence and its consequences for effective field theories are analyzed in the next couple of sections. We close with a summary and concluding remarks. The Appendix section elaborates on the topic of transversality constraints in variational problems and their connection to the issue of sensitive dependence.

2. Stationary action in classical field theory

Classical field theory develops from the Lagrangian

$$L = L(\varphi, \partial_\mu \varphi, x^\mu) \tag{1}$$

and the first order variation of the action functional given by [10-11]

$$\delta S = \int_R \delta\varphi\, d^4x \{\frac{\partial L}{\partial \varphi} - \partial_\mu [\frac{\partial L}{\partial(\partial_\mu \varphi)}]\} + \int_{\partial R} d\sigma_\mu [\frac{\partial L}{\partial(\partial_\mu \varphi)}\delta\varphi + L\delta x^\mu] \tag{2}$$

Here, R is the four-dimensional integration domain whose boundary is ∂R. The canonical treatment of (2) posits that both field and coordinate variations vanish on ∂R, i.e.

$$\delta\varphi = \delta x^\mu = 0 \;\text{ on }\; \partial R \tag{3}$$

which supplies the field equation in the standard form

$$\frac{\delta S[\varphi]}{\delta \varphi} = \Lambda(\varphi, \partial_\mu \varphi, x^\mu) = \frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x^\mu}[\frac{\partial L}{\partial(\partial_\mu \varphi)}] = 0 \tag{4}$$

Applying (2) to classical electrodynamics ($\varphi \to A$) and General Relativity, respectively, ($\varphi \to g$), yields

$$\int d^4x\,\delta A_\mu (j^\mu - \frac{\partial F^{\mu\nu}}{\partial x^\nu}) = 0 \tag{5}$$

$$\int d^4x\sqrt{-g}\,\delta g^{\mu\nu}[\frac{1}{2}T_{\mu\nu} - \frac{1}{16\pi G}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R)] = 0 \tag{6}$$

(5) and (6) lead to Maxwell and Einstein equations under the textbook setting that the field variations δA_μ and $\delta g^{\mu\nu}$ are arbitrary and the functional differential equation (4) is satisfied [10-11, 13].

2. Dependent endpoint conditions

Consider the generic case where fields and spacetime coordinates vary simultaneously on R, while the boundary term goes to zero at infinity. If the first term of (2) is sampled at fixed time intervals δt, a convenient approximation of δS over a discrete set of sampling points $i = 1, 2, ..., N$ can be written as

$$\delta S = \int_R d^4x\, d\varphi\, \Lambda \propto \delta t [\sum_i \delta^3 x_i\, \delta\varphi_i\, \Lambda_i] \tag{7}$$

where

$$\delta x_1 << 1, \delta\varphi_1 << 1;\ N >> 1 \tag{8a}$$

$$\varphi_i = \varphi(x_i) \tag{8b}$$

$$\Lambda_i = \Lambda(\varphi_i, \partial^x \varphi_i, x_i) \tag{8c}$$

Sensitivity to initial conditions in coordinate and field spaces, respectively, causes exponential separation of adjacent trajectories as in

$$\delta\varphi_{i+1}(x_{i+1}) \propto \delta\varphi_i(x_i)\exp[\lambda(\varphi_{i+1} - \varphi_i)]; \quad \lambda > 0; \quad i > 1 \tag{9}$$

and

$$\delta x_{i+1} \propto \delta x_i \exp[\sigma(x_{i+1} - x_i)]; \quad \sigma > 0; \quad i > 1 \tag{10}$$

Coordinates and fields can only be measured to finite precision. This is to say that, in fact, there are *infinitely many* adjacent trajectories defined through

$$x_1 = x_1 + |\mathrm{X}| \tag{11a}$$

$$\varphi_1 = \varphi_1 + |\Omega| \tag{11b}$$

with uncertainties upper bounded by their *resolution limits* respectively, that is,

$$|X| \leq R_x \tag{12a}$$

$$|\Omega| \leq R_\varphi \tag{12b}$$

Conditions (11) and (12) imply that all adjacent trajectories starting from points located within R_x and/or R_φ are initially *indistinguishable* from each other, even though they split apart later on. It follows that the endpoint variations of both field and coordinates are no longer independent and likely to become ill-defined for sufficiently large separations ($x_{i+1} >> x_i$ and $\varphi_{i+1} >> \varphi_i$). Stated differently, dependent endpoint conditions induce *memory-like effects* and are asymptotically *unpredictable.* Another way to look at these observations is to acknowledge that deterministic dynamics of classical field theory no longer stands, in manifest contrast with the foundation of Maxwell's electrodynamics and General Relativity.

Although somewhat unexpected, these findings are nevertheless hardly surprising. They merely confirm the long-held belief that classical field theories complying with (3) are *effective field approximations,* endowed with limited ranges of validity.

Similar conclusions apply to situations where non-holonomic constraints and/or the virial theorem need to be accounted for [14-15]. For the sake of simplicity and concision, these cases are not included here.

…*text to follow*…

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